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The pseudospectrum of the convention-diffusion operator with a variable reaction term

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Псевдоспектр оператора конвенции-диффузии с переменным членом реакции

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Abstract. In this paper, we study the spectrum of non-self-adjoint convection-diffusion operator with a variable reaction term defined on an unbounded open set Ω of $\mathbb{R}^n$. Our idea is to build a family of operators that have the same convection-diffusion-reaction formula, but which will be defined on bounded open sets $\{\Omega_\eta\}_{\eta\in]0,1[}$ of $\mathbb{R}^n$. Based on the relationships that link this family to Ω, we obtain relations between the spectrum and the pseudospectrum. We use the notion of the pseudospectrum to build relationships between convection-diffusion operator and its restrictions to bounded domains. Using these relationships we are able to find the spectrum of our operator in $\mathbb{R}^+$. Also, the techniques developed to obtain the spectrum allow us to study the properties of the spectrum of this operator when we go to the limit as the reaction term tends to zero. Indeed, we show a spectral localization result for the same convection-diffusion-reaction operator when a perturbation is carried on the reaction term and no longer on the definition domain.

Keywords: differential operator; spectrum; pseudospectrum; convention-diffusion operator

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Аннотация. В статье исследуется спектр несамосопряженного оператора конвекции-диффузии с переменным членом реакции, определенным на неограниченном открытом множестве $\Omega \subset \mathbb{R}^n$. Идея исследования состоит в том, чтобы построить семейство операторов, имеющих такую же формулу конвекции-диффузии-реакции, но определенных на ограниченных открытых множествах $\{\Omega_\eta\}_{\eta\in]0,1[} \subset \mathbb{R}^n$. Основываясь на соотношениях, которые связывают это семейство с Ω, получены соотношения между спектром и псевдоспектром. Для построения соотношений между оператором конвекции-диффузии и его сужениями на ограниченные области используется понятие псевдоспектра. Полученные соотношения используются для определения спектра исходного оператора в $\mathbb{R}^+$. Методы, разработанные для нахождения спектра заданного оператора, позволяют также изучить некоторые свойства этого спектра при переходе к пределу, когда член реакции стремится к нулю. В частности, показано, как определить спектр заданного оператора конвекции-диффузии-реакции при возмущении члена реакции, а не области определения.

Ключевые слова: дифференциальный оператор; спектр; псевдоспектр; оператор конвенции-диффузии

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Introduction

The study of the spectrum of convection-diffusion operator is one of the most complicated problems in functional analysis. In this paper, we study the spectrum of the following operator:

$$Au = -\triangle u + \left(- \nabla h(\prod_{i=1}^{n} x_i) \right) \cdot \nabla u + Vu, \tag{0.1}$$

where $h \in C^2(\mathbb{R}, \mathbb{R})$ such that h'' is positive and

$$V = V_1 + V_2, \quad V_1(x) = \sum_{i=1}^{n} \left(h'(\prod_{j=1}^{n} x_j) \prod_{\substack{j=1 \\ j\neq i}}^{n} x_i \right)^2.$$

Non-negative potential V_2 is considered as a reaction to the convection-diffusion phenomena represented by $A_0 = -\Delta + \left(- \nabla h(\prod_{i=1}^{n} x_i) \right) \cdot \nabla + V_1$. The operator A in the case $h(x) = x$ was studied in [1].

We recall that, for an unbounded operator T defined on $D(T) \subset H$ to H and for $\varepsilon > 0$, the pseudospectrum is given by (see [2])

$$sp_\varepsilon(T) = \left\{ z \in re(T) \ : \ \left\| (zI - T)^{-1} \right\| > \varepsilon^{-1} \right\} \bigcup sp(T).$$

The resolvent set is given by

$$re(T) = \left\{ z \in \mathbb{C} \ : \ (zI - T)^{-1} \text{ exists and bounded} \right\};$$

$sp(T)$ denotes the spectrum of T and is defined as $sp(T) = \mathbb{C} \setminus re(T)$. An equivalent definition of the pseudospectrum has been given in [3]:

$$sp_\varepsilon(T) = \bigcup_{D: H \to H, \ linear \ and \ \|D\| < \varepsilon} sp(T + D).$$

Pseudospectrum is easier to calculate and more efficient than spectrum when dealing with unbounded operators [4, 5]. In fact, it has been established that an approximation of the spectrum of differential operators may be unstable when going to the limit, unlike the pseudospectrum which shows to be stable (see [6–8]). For example, if T is a normal operator, its pseudospectrum is equal to the ε-neighborhood of its spectrum. The ε-neighborhood of $S \subset \mathbb{C}$ is given by

$$N_\varepsilon(S) = \{s + z \ : \ s \in S, \, |z| < \varepsilon\}.$$

It is clear that, for all $S \subset \mathbb{C}$, $\bigcap_{\varepsilon>0} N_\varepsilon(S) = S$.

Moreover we take advantage of the fact that the spectrum of an operator is divided into two sets: the pointwise spectrum, $sp_{\mathrm{p}}(T)$, which consists of all the eigenvalues of T; the essential spectrum, $sp_{\mathrm{ess}}(T)$ which consists of all $\lambda \in \mathbb{C}$ such that the operator $(\lambda I - T)$ is injective, but not surjective. In addition, we define the limit of a sequence of sets as follows: for all $\theta > 0$, $S_\theta \subset \mathbb{C}$, $\lim_{\theta\to 0} S_\theta = \left\{s \in \mathbb{C} \ : \ \exists \{s_\theta\}_{\theta>0}, \ s_\theta \in S_\theta, \ \lim_{\theta \to 0} s_\theta = s\right\}$.

In this article, we study in detail the pseudospectrum and the spectrum of the operator A to establish that its spectrum is real positive. We conclude this work with a result on the stability of the spectrum obtained by the pseudospectral theory.

1. Convection-diffusion operator

Let $\Omega \subsetneq \mathbb{R}^n$ be an unbounded open set. Let $h \in C^2(\mathbb{R}, \mathbb{R})$ be such that h'' is positive. Let A be the convection-diffusion operator (see [9]) defined on $L^2(\Omega, \mathbb{C})$ into itself by (0.1).

We define the hermitian form φ on $L^2(\Omega, \mathbb{C})$ as

$$\varphi(f,g) = \int_\Omega \nabla f \cdot \overline{\nabla g} dx + \int_\Omega \Big(-\nabla h(\prod_{i=1}^n x_i)\Big) \cdot \nabla f \overline{g} dx + \int_\Omega V f \overline{g} dx,$$

where the quadratic form associated with φ is given by

$$Q(u) = \|\nabla u\|^2_{L^2(\Omega)} + \int_\Omega \Big(-\nabla h(\prod_{i=1}^n x_i)\Big) \cdot \nabla u \overline{u} dx + \int_\Omega V |u|^2 dx.$$

For $u \in C_c^\infty(\Omega)$, we set $z = \int_\Omega (-\nabla h(\prod_{i=1}^n x_i))\, \overline{u} \cdot \nabla u dx$, so

$$\begin{aligned} z &= \underbrace{\int_{\partial\Omega} \Big(-\nabla h(\prod_{i=1}^n x_i)\Big) |u|^2 dx}_{=0} + \int_\Omega \Big(\Delta h(\prod_{i=1}^n x_i)\Big) |u|^2 dx \\ &+ \int_\Omega \Big(\nabla h(\prod_{i=1}^n x_i)\Big) \cdot \overline{\nabla u} u dx = \int_\Omega \Big(\Delta h(\prod_{i=1}^n x_i)\Big) |u|^2 dx - \overline{z}. \end{aligned}$$

Using $z + \overline{z} = 2Re(z) = \int_\Omega \big(\Delta h(\prod_{i=1}^n x_i)\big)|u|^2 dx$ yields

$$Re(Q(u)) = \|\nabla u\|^2_{L^2(\Omega)} + \int_\Omega \Big(\frac{1}{2}\Delta h(\prod_{i=1}^n x_i) + \|\nabla h(\prod_{i=1}^n x_i)\|^2_{L^2(\Omega)} + V_2(x)\Big) |u|^2 dx \geq 0,$$

and

$$|Im(Q(u))| = \Big|Im\Big(\int_{\Omega}\Big(-\nabla h(\prod_{i=1}^{n}x_i)\Big)\overline{u}\cdot\nabla u dx\Big)\Big| = \Big|\frac{z-Re(z)}{i}\Big|$$
$$\leq \frac{1}{2}\Big(\|\nabla u\|^2_{L^2(\Omega)} + \int_{\Omega}\|\nabla h(\prod_{i=1}^{n}x_i)\|^2_{L^2(\Omega)}|u|^2dx\Big) + \frac{1}{2}\Big|\int_{\Omega}\Big(\Delta h(\prod_{i=1}^{n}x_i)\Big)|u|^2dx\Big|.$$

Hence, φ is a sectorial form defined on the vector space R given by the following expression:

$$R = H^1_0(\Omega,\mathbb{C})\bigcap\left\{u\in L^2(\Omega,\mathbb{C}) : Vu\in L^2(\Omega,\mathbb{C})\right\}.$$

We recall that A is the operator associated with φ [10, Theorem 2.1, p. 322], and the domain of A is given by $D(A) = H^2(\Omega,\mathbb{C})\bigcap R$.

Our goal is to determine the spectrum of A. We notice that $D(A)$ is a Dirichlet integral boundary condition. Consider the eigenvalue problem: *find* $\lambda\in\mathbb{C}$ *and* $u\in D(A)\setminus\{0\}$ *such that*

$$-\Delta u + \Big(-\nabla h(\prod_{i=1}^{n}x_i)\Big)\cdot\nabla u + Vu = \lambda u \text{ on } \Omega,\quad u=0 \text{ on } \partial\Omega.$$

Let $\{\Omega_\eta\}_{\eta\in]0,1[}$ be a sequence of bounded open sets such that $\Omega_{\acute{\eta}}\subset\Omega_\eta$, for all $\eta\leq\acute{\eta}$, and $\bigcup_{\eta\in]0,1[}\Omega_\eta = \Omega$. For all $\eta\in]0,1[$, we identify the hermitian form φ_η on $L^2(\Omega_\eta,\mathbb{C})$ by

$$\varphi_\eta(f,g) = \int_{\Omega_\eta}\nabla f\cdot\overline{\nabla g}dx + \int_{\Omega_\eta}\left(-\nabla h(\prod_{i=1}^{n}x_i)\right)\cdot\nabla f\overline{g}dx + \int_{\Omega_\eta}Vf\overline{g}dx.$$

We recall that φ_η is a sectorial form defined on $R_\eta = H^1_0(\Omega_\eta,\mathbb{C})$. We denote by A_η the differential operator associated with φ_η [10, Theorem 2.1, p. 322]. The domain of A_η is given by $D(A_\eta) = H^2(\Omega_\eta,\mathbb{C})\bigcap H^1_0(\Omega_\eta,\mathbb{C})$. We notice that A_η is defined by the same formula as A. Let B_η be the differential operator which is defined by the same formula as A, but is given on $D(B_\eta) = H^2_0(\Omega_\eta,\mathbb{C})$. The extension of each function in $D(B_{\eta'})$ to Ω_η by zero belongs to $D(B_\eta)$ [11, Lemma 3.22, p. 57]. Thus $D(B_{\eta'})\subset D(B_\eta)$.

To achieve our goal we will define the spectrum of A_η, after that we will establish a relation between the pseudospectrum and spectrum of A_η, B_η and A.

2. Spectrum of A_η and B_η

In this section, we will explain some characteristics of the operator A_η which allow us to locate the spectrum of A.

2.1. Spectrum of A_η

Define the following scalar product on $L^2(\Omega_\eta)$:

$$\forall(u,v)\in L^2(\Omega_\eta)\times L^2(\Omega_\eta),\quad \langle u,v\rangle_\eta = \int_{\Omega_\eta}\exp\Big(h(\prod_{i=1}^{n}x_i)\Big)u\overline{v}dx.$$

Theorem 2.1. *For all* $\eta\in]0,1[$, A_η *is self-adjoint with respect to* $\langle\,,\,\rangle_\eta$.

P r o o f. For all $u \in D(A_\eta)$, we define $\widetilde{u} = \exp\Big(\frac{1}{2}h(\prod_{i=1}^{n} x_i)\Big)u$. Hence, we get

$$\Delta\widetilde{u} = \Big(\Delta u + \Big(\nabla h(\prod_{i=1}^{n} x_i)) \cdot \nabla u\Big) + \frac{1}{4}\|\nabla h(\prod_{i=1}^{n} x_i)\|^2_{L^2(\Omega_\eta)} u\Big) \exp\Big(\frac{1}{2}h(\prod_{i=1}^{n} x_i)\Big).$$

Then, for all $(u,v) \in D(A_\eta) \times D(A_\eta)$,

$$\begin{aligned}
\langle A_\eta u, v\rangle_\eta &= \int_{\Omega_\eta} \exp\Big(h(\prod_{i=1}^{n} x_i)\Big) A_\eta u\overline{v}dx = \int_{\Omega_\eta} \Big(- \Delta u + \big(- \nabla h(\prod_{i=1}^{n} x_i)\big) \cdot \nabla u + \\
&\quad + \Big(\|\nabla h(\prod_{i=1}^{n} x_i)\|^2_{L^2(\Omega_\eta)} + V_2(x)\Big)u\Big) \exp\left(\frac{1}{2}h(\prod_{i=1}^{n} x_i)\right) \overline{\widetilde{v}}dx \\
&= \int_{\Omega_\eta} -\Delta\widetilde{u}\overline{\widetilde{v}}dx + \int_{\Omega_\eta} \Big(\frac{5}{4}\|\nabla h(\prod_{i=1}^{n} x_i)\|^2_{L^2(\Omega_\eta)} + V_2(x)\Big)\widetilde{u}\overline{\widetilde{v}}dx \\
&= \int_{\Omega_\eta} \nabla\widetilde{u}\overline{\nabla\widetilde{v}}dx - \int_{\partial\Omega_\eta} \frac{\partial\widetilde{u}}{\partial v} \underbrace{\overline{\widetilde{v}}}_{=0} d\sigma + \int_{\Omega_\eta} \Big(\frac{5}{4}\|\nabla h(\prod_{i=1}^{n} x_i)\|^2_{L^2(\Omega_\eta)} + V_2(x)\Big)\widetilde{u}\overline{\widetilde{v}}dx.
\end{aligned}$$

So, for all $\eta \in]0,1[$, $\langle A_\eta\cdot,\cdot\rangle_\eta$ is also a scalar product, which means that A_η is self-adjoint [10, Theorem 2.7]. □

As a consequence, $sp(A_\eta)$ is real for all $\eta \in]0,1[$. Since we cannot extend the scalar product $\langle\, ,\, \rangle_\eta$ over $L^2(\Omega)$, we cannot guarantee that A is self-adjoint.

We define

$$\begin{aligned}
K &= \inf\Big\{\frac{1}{2}\Delta h(\prod_{i=1}^{n} x_i) + \|\nabla h(\prod_{i=1}^{n} x_i)\|^2_{L^2_{\Omega_\eta}} + V_2(x) \;:\; (x_1,....,x_n) \in \Omega_\eta\Big\}, \\
K_1 &= \inf\Big\{\|\nabla h(\prod_{i=1}^{n} x_i)\|^2_{L^2_{\Omega_\eta}} \;:\; (x_1,....,x_n) \in \Omega_\eta\Big\}, \\
M &= C_{PF}^{-2} + K - \frac{5K_1}{4}, \\
E &= \{\eta \in]0,1] \;:\; M \leq 0\},
\end{aligned}$$

where C_{PF} is the Poincarre-Friedrichs constant (see [10]).

Theorem 2.2. *For all* $\eta \in]0,1[$,

- *If* $\eta \notin E$, *the essential spectrum* $sp_{ess}(A_\eta)$ *is included in* $]\frac{5}{4}K_1, C_{PF}^{-2} + K[$, *and the point spectrum* $sp_p(A_\eta)$ *is included in* $[C_{PF}^{-2} + K, +\infty[$,
- *If* $\eta \in E$, A_η *has no essential spectrum, and the point spectrum is included in* $[C_{PF}^{-2} + K, +\infty[$.

P r o o f. For all $\eta \in]0,1[$ and all $u \in D(A_\eta)$,

$$Re(\langle A_\eta u, u\rangle) = \frac{1}{2}(\langle A_\eta u, u\rangle + \overline{\langle u, A_\eta u\rangle}) = \frac{1}{2}(\langle A_\eta u, u\rangle + \langle u, A_\eta u\rangle).$$

However

$$\begin{aligned}\int_{\Omega_\eta} -\Delta u\overline{u}dx &= -\int_{\partial\Omega_\eta}\frac{\partial u}{\partial\nu}\underbrace{\overline{u}}_{=0}d\sigma + \int_{\Omega_\eta}\nabla u\nabla\overline{u}dx = \int_{\Omega_\eta}|\nabla u|^2dx.\\ \int_{\Omega_\eta}\nabla h(\prod_{i=1}^n x_i)\cdot\nabla u\overline{u}dx &= -\int_{\Omega_\eta}\nabla h(\prod_{i=1}^n x_i)\cdot\overline{\nabla u}udx - \int_{\Omega_\eta}\Delta h(\prod_{i=1}^n x_i)\,|u|^2\,dx.\end{aligned}$$

Then

$$\begin{aligned}Re(\langle A_\eta u,u\rangle) &= \|\nabla u\|^2_{L^2(\Omega_\eta)} + \int_{\Omega_\eta}\left(\frac{1}{2}\Delta h(\prod_{i=1}^n x_i) + \|\nabla h(\prod_{i=1}^n x_i)\|^2_{L^2(\Omega_\eta)} + V_2(x)\right)|u|^2dx\\ &\geq \|\nabla u\|^2_{L^2(\Omega_\eta)} + K\|u\|^2_{L^2(\Omega_\eta)}.\end{aligned}$$

By the theorem of Poincarre-Friedrich (see [10]),

$$Re(\langle A_\eta u,u\rangle) \;\geq\; C_{PF}^{-2}\|u\|^2_{L^2(\Omega_\eta)} + K\|u\|^2_{L^2(\Omega_\eta)} \geq (C_{PF}^{-2}+K)\|u\|^2_{L^2(\Omega_\eta)}.$$

Thus, for all $\lambda\in\mathbb{R}$,

$$\|(A_\eta-\lambda I)u\|_{L^2(\Omega_\eta)} \geq (C_{PF}^{-2}+K-\lambda)\|u\|_{L^2(\Omega_\eta)}.$$

Hence $(A_\eta-\lambda I)$ is injective for all $\lambda < C_{PF}^{-2}+K$, and thus $sp(A_\eta)$ is included in $[C_{PF}^{-2}+K,+\infty[$. Let $H=H_0^1(\Omega_\eta)$ and $\lambda\in[-\infty,\frac{5}{4}K_1[$. The sesquilinear form is defined on H by

$$\varphi_\lambda(u,v) = \int_{\Omega_\eta}(\nabla u\nabla\overline{v} + (\frac{5}{4}V_1+V_2-\lambda)u\overline{v})dx,$$

which verifies

$$|\varphi_\lambda(u,v)| \leq \|\nabla u\|_{L^2(\Omega_\eta)}\|\nabla v\|_{L^2(\Omega_\eta)} + C\|u\|_{L^2(\Omega_\eta)}\|v\|_{L^2(\Omega_\eta)},$$

where

$$C = \sup\left\{\frac{5}{4}V_1(x)+V_2(x)\;:\;x\in\Omega_\eta\right\} + |\lambda|,$$

and

$$|\varphi_\lambda(u,v)| \geq \min\left\{1,(\frac{5}{4}K_1-\lambda)\right\}\|u\|^2_H.$$

Since, for all $g\in L^2(\Omega_\eta)$, the semilinear form $L: H\to\mathbb{C},\ v\mapsto\int_{\Omega_\eta}g\overline{v}dx$ is continuous, it follows from the Lax-Miligram theorem that the equation

$$\varphi_\lambda(u,v) = L(v)$$

has a unique solution u in H for all $v\in H$. Take into consideration the problem

$$(p)\begin{cases}\text{for } g\in L^2(\Omega_\eta), & \text{find } u\in L^2(\Omega_\eta) \text{ such that}\\ A_\eta u-\lambda u = g & \text{on } \Omega_\eta,\\ u=0 & \text{on } \partial\Omega_\eta.\end{cases}$$

We use the same variable change as in the previous theorem: we multiply the equation by $\exp(\frac{h(\prod_{i=1}^n x_i)}{2})$ and set $\widetilde{g} = g\exp(\frac{h(\prod_{i=1}^n x_i)}{2})$, $\widetilde{u} = u\exp(\frac{h(\prod_{i=1}^n x_i)}{2})$. We see that (p) is equivalent to

$$(\widetilde{p}) \begin{cases} \text{for } \widetilde{g} \in L^2(\Omega_\eta), & \text{find } \widetilde{u} \in L^2(\Omega_\eta) \text{ such that} \\ -\Delta\widetilde{u} + (\frac{5}{4}V_1 + V_2)\widetilde{u} - \lambda\widetilde{u} = \widetilde{g} & \text{on } \Omega_\eta, \\ \widetilde{u} = 0 & \text{on } \partial\Omega_\eta. \end{cases}$$

Therefore, the sesquilinear form $\varphi_\lambda(u,v) = \int_{\Omega_\eta} \nabla u \nabla \overline{v} + (\frac{5}{4}K_1 - \lambda)u\overline{v})$ is an inner product in $L^2(\Omega_\eta)$ for $\lambda < \frac{5}{4}K_1$. Hence, $(\widetilde{p})$ has a unique solution $\widetilde{u}$, and (p) has a unique solution u defined by $u = \widetilde{u}\exp\big(-h\big(\prod_{i=1}^n x_i\big)\big)$. □

2.2. Relation between A_η and B_η

It is known that, for every $\eta \in]0,1[$, $B_\eta \subseteq A_\eta$, i. e. $D(B_\eta) \subseteq D(A_\eta)$ and, for all $f \in D(B_\eta)$,

$$B_\eta f = A_\eta f.$$

Furthermore, for every $\eta \in]0,1[$, $B_\eta \subseteq A$. In fact, for any $f \in D(B_\eta) = H_0^2(\Omega_\eta)$, extending f to Ω by 0 gives $f \in D(A)$. This proves that

$$\bigcup_{0<\eta<1} sp_p(B_\eta) \subseteq sp_p(A_\eta).$$

To determine the spectrum of A the study of the difference $sp_p(A_\eta)\backslash sp_p(B_\eta)$ is required. In this section, we prove that this difference is empty for $\eta \in]0,1[$.

Lemma 2.1. *For all* $\varepsilon > 0$ *and for all* $\eta \in]0,1[$,

$$sp_\varepsilon(B_\eta) = sp_\varepsilon(A_\eta).$$

P r o o f. Let $\lambda \in sp_\varepsilon(B_\eta)$. Then there exists $f \in D(B_\eta)$ such that

$$\frac{\|B_\eta f - \lambda f\|_{L^2(\Omega_\eta)}}{\|f\|_{L^2(\Omega_\eta)}} < \varepsilon.$$

But $f \in D(A_\eta)$, so

$$\frac{\|A_\eta f - \lambda f\|_{L^2(\Omega_\eta)}}{\|f\|_{L^2(\Omega_\eta)}} < \varepsilon,$$

and $\lambda \in sp_\varepsilon(A_\eta)$.

Inversely, let $\lambda \in sp_\varepsilon(A_\eta)$. Then there is $f \in D(A_\eta)$ such that

$$\frac{\|A_\eta f - \lambda f\|_{L^2(\Omega_\eta)}}{\|f\|_{L^2(\Omega_\eta)}} < \varepsilon.$$

Since the space of infinitely differentiable functions with compact support $C_c^\infty(\Omega_\eta)$ is dense in $D(A)$ with respect to the graph norm defined by $\|\cdot\|_A = \|A\cdot\|_{L^2(\Omega)} + \|\cdot\|_{L^2(\Omega)}$, for all

$f \in D(A_\eta)$, there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\Omega_\eta)$ such that $\lim_{n\to+\infty} \|f_n - f\|_A = 0$. So, for all $\theta > 0$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, we have

$$\left| \frac{\|A_\eta f_n - \lambda f_n\|_{L^2(\Omega)}}{\|f_n\|_{L^2(\Omega)}} - \frac{\|A_\eta f - \lambda f\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}} \right| < \theta.$$

We set

$$\theta = \varepsilon - \frac{\|A_\eta f - \lambda f\|_{L^2(\Omega_\eta)}}{\|f\|_{L^2(\Omega_\eta)}} > 0.$$

Then there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\|A_\eta f_{n_0} - \lambda f_{n_0}\|_{L^2(\Omega)}}{\|f_{n_0}\|_{L^2(\Omega)}} < \varepsilon.$$

However $f_{n_0} \in D(B_\eta)$, then $\lambda \in sp_\varepsilon(B_n)$. □

Corollary 2.1. *For all* $\varepsilon > 0$, *if* $0 < \eta \leq \eta' < 1$, *then*

$$sp_\varepsilon(A_{\eta'}) \subseteq sp_\varepsilon(A_\eta).$$

P r o o f. Let $\lambda \in sp_\varepsilon(B_{\eta'})$. Then there exists $f \in D(B_{\eta'})$ such that

$$\frac{\|B_{\eta'} f - \lambda f\|_{L^2(\Omega_{\eta'})}}{\|f\|_{L^2(\Omega_{\eta'})}} < \varepsilon.$$

Extending f to Ω_η by 0,

$$\frac{\|B_\eta f - \lambda f\|_{L^2(\Omega_\eta)}}{\|f\|_{L^2(\Omega_\eta)}} < \varepsilon.$$

It follows that λ belongs to $sp_\varepsilon(B_\eta)$. Now we can apply Lemma 2.1 to complete the proof. □

We proved that the family $\{sp_\varepsilon(A_\eta)\}_{0<\eta<1}$ is decreasing, and this makes us to say that the family $\{sp(A_\eta)\}_{0<\eta<1}$ is decreasing with respect to inclusion. In fact, for all $0 < \eta \leq \eta' < 1$,

$$sp(A_{\eta'}) \subseteq sp_\varepsilon(A_{\eta'}) \subseteq sp_\varepsilon(A_\eta).$$

But A_η is self-adjoint, i. e. $sp_\varepsilon(A_\eta) = N_\varepsilon\left(sp(A_\eta)\right)$, for all $\eta \in]0,1[$. Then

$$sp(A_{\eta'}) \subset \bigcap_{\varepsilon>0} N_\varepsilon\left(sp(A_\eta)\right) = sp(A_\eta).$$

Theorem 2.3. *For all* $\eta \in E$,

$$sp(B_\eta) = sp(A_\eta).$$

P r o o f. Since A_η is self-adjoint for all $\eta \in E$, we have

$$\bigcap_{\varepsilon>0} sp_\varepsilon(A_\eta) = sp(A_\eta).$$

Then, by using Lemma 2.1, we find

$$sp(B_\eta) \subseteq sp_\varepsilon(B_\eta) = sp_\varepsilon(A_\eta) \Rightarrow sp(B_\eta) \subseteq \bigcap_{\varepsilon>0} sp_\varepsilon(A_\eta) = sp(A_\eta).$$

Reciprocally, let $\lambda \in sp(A_\eta)$ for some $\eta \in E$. According to Theorem 2.2, there exists $f \in D(A), f \neq 0$ such that

$$A_\eta f = \lambda f.$$

Since $C_c^\infty(\Omega_\eta)$ is dense in $D(A_\eta)$ with respect to the graph norm, there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\Omega_\eta)$ which converges to f in the graph norm. We define the sequence

$$g_n = \frac{f_n}{\|f_n\|_{L^2(\Omega_\eta)}}, \qquad n \in \mathbb{N}.$$

For all $n \in \mathbb{N}$,

$$g_n \in D(B_\eta), \qquad \|g_n\|_{L^2(\Omega_\eta)} = 1,$$

and

$$\lim_{n\to+\infty} \|B_\eta g_n - \lambda g_n\|_{L^2(\Omega_\eta)} = \frac{\|A_\eta f - \lambda f\|_{L^2(\Omega_\eta)}}{\|f\|_{L^2(\Omega_\eta)}} = 0.$$

Then $\lambda \in sp(B_\eta)$. In fact, if $(B_\eta - \lambda I)^{-1}$ exists and is bounded, then

$$1 = \|g_n\|_{L^2(\Omega_\eta)} \leq \left\|(B_\eta - \lambda I)^{-1}\right\| \|B_\eta g_n - \lambda g_n\|_{L^2(\Omega_\eta)} \longrightarrow 0.$$

Therefor, the operator $(B_\eta - \lambda I)^{-1}$, if it exists, can not be bounded, which means that $B_\eta - \lambda I$ can not be surjective. □

3. Pseudospectrum and spectrum of A

The pseudospectrum has better stability than the spectrum. Pseudospectrum is easier to be controlled and can be considered as the finest stable for the passage to the limit.

3.1. Pseudospectrum

In this subsection, we establish a relation between the spectrum and the pseudospectrum of A seen as limits of $sp\left(A_\eta\right)$ and $sp\left(B_\eta\right)$ respectively.

Theorem 3.1. *For all* $\varepsilon > 0$,

$$sp_\varepsilon(A) = \bigcup_{\eta\in E} sp_\varepsilon(A_\eta) = \bigcup_{\eta\in E} sp_\varepsilon(B_\eta).$$

P r o o f. Let $\lambda \in \bigcup_{0<\eta<1} sp_\varepsilon(B_\eta)$. Then there exist $\eta_1 \in E$ and $f \in D(B_{\eta_1})$ such that

$$\frac{\|B_{\eta_1} f - \lambda f\|_{L^2(\Omega_{\eta_1})}}{\|f\|_{L^2(\Omega_\eta)}} < \varepsilon.$$

Extending f to Ω by 0,

$$\frac{\|Af-\lambda f\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}}<\varepsilon.$$

So, it follows that λ belongs to $sp_\varepsilon(A)$, and

$$\bigcup_{\eta\in E} sp_\varepsilon(B_\eta)\subseteq sp_\varepsilon(A).$$

Reciprocally, let $\lambda\in sp_\varepsilon(A)$. Then there is $f\in D(A)$ such that

$$\frac{\|Af-\lambda f\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}}<\varepsilon.$$

Since $C_c^\infty(\Omega)$ is dense in $D(A)$ with respect to the graph norm, for all $f\in D(A)$, there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\Omega)$ such that

$$\lim_{n\to+\infty}\frac{\|Af_n-\lambda f_n\|_{L^2(\Omega)}}{\|f_n\|_{L^2(\Omega)}}=\frac{\|Af-\lambda f\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}}.$$

Like in the proof of Lemma 2.1, we choose n_0 such that

$$\frac{\|Af_{n_0}-\lambda f_{n_0}\|_{L^2(\Omega)}}{\|f_{n_0}\|_{L^2(\Omega)}}<\varepsilon.$$

There is η small enough for which the support of $g=f_{n_0}$ is included in Ω_η. It follows that λ belongs to $sp_\varepsilon(A_\eta)$. Thus,

$$sp_\varepsilon(A)\subseteq\bigcup_{\eta\in E} sp_\varepsilon(A_\eta).$$

Now, we use Lemma 2.1 to conclude the proof. □

From the previous theorem, we deduce that

$$sp_\varepsilon(A)\subseteq N_\varepsilon(\mathbb{R}^+)=\{z\in\mathbb{C}: Rez>0, |Imz|<\varepsilon\}\bigcup\{z\in\mathbb{C}: Rez<0, |z|<\varepsilon\}.$$

In fact, for all $\eta\in E$, A_η is self-adjoint. Then $sp_\varepsilon(A_\eta)=N_\varepsilon(sp(A_\eta))$. But, $sp(A_\eta)\subseteq\mathbb{R}^+$. We obtain

$$\bigcup_{\eta\in E} sp(A_\eta)\subseteq\mathbb{R}^+.$$

3.2. Spectra

In this part, we will set a new relation between the spectrum of A and that of A_η. First, we begin with a topological result that will allow us to obtain the desired property.

P r o p o s i t i o n 3.1. *For all* $\varepsilon>0$,

$$\bigcup_{\eta\in E} sp_\varepsilon(A_\eta)=N_\varepsilon\left(\bigcup_{\eta\in E} sp(A_\eta)\right).$$

P r o o f. Let $\lambda \in \bigcup_{\eta\in E} sp_\varepsilon(A_\eta)$. There is $\eta_1 \in E$ such that

$$\lambda \in sp_\varepsilon(A_{\eta_1}) = N_\varepsilon\left(sp(A_{\eta_1})\right).$$

So, $\lambda = s + z$, where $s \in sp(A_{\eta_1})$ and $|z| < \varepsilon$. But $s \in \bigcup_{\eta\in E} sp(A_\eta)$ implies

$$\lambda \in N_\varepsilon\left(\bigcup_{\eta\in E} sp(A_\eta)\right).$$

Reciprocally, let $\lambda \in N_\varepsilon\left(\bigcup_{\eta\in E} sp(A_\eta)\right)$. Then $\lambda = s+z$, where $s \in \bigcup_{\eta\in E} sp(A_\eta)$ and $|z| < \varepsilon$. So, there is $\eta_1 \in E$ such that

$$\lambda = s + z \in N_\varepsilon(sp(A_{\eta_1})) = sp_\varepsilon(A_{\eta_1}).$$

Thus, $\lambda \in \bigcup_{\eta\in E} sp_\varepsilon(A_\eta)$. □

Theorem 3.2.

$$sp(A) = \bigcup_{\eta\in E} sp(A_\eta).$$

P r o o f. Let $\lambda \in \bigcup_{\eta\in E} sp(A_\eta)$. There is $\eta_1 \in E$ such that $\lambda \in sp(A_{\eta_1})$. Then there is $f \in D(A_{\eta_1})$, where $A_{\eta_1} f - \lambda f = 0$. Since $C_c^\infty(\Omega_{\eta_1})$ is dense in $D(A_{\eta_1})$ with respect to the graph norm, there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\Omega_{\eta_1})$ which converges to f in the graph norm. We define the sequence

$$g_n = \begin{cases} \frac{f_n}{\|f_n\|_{L^2(\Omega_\eta)}} & \text{on } \Omega_{\eta_1}, \quad n \in \mathbb{N}, \\ 0 & \text{on } \Omega/\Omega_{\eta_1}, \quad n \in \mathbb{N}. \end{cases}$$

We have, for all $n \in \mathbb{N}$,

$$g_n \in D(A), \|g_n\|_{L^2(\Omega_\eta)} = 1,$$

then

$$\lim_{n\to+\infty} \|Ag_n - \lambda g_n\|_{L^2(\Omega_\eta)} = 0.$$

Thus $\lambda \in sp(A)$. By Theorem 3.1, we have

$$sp(A) \subset sp_\varepsilon(A) = \bigcup_{\eta\in E} sp_\varepsilon(A_\eta),$$

and by Proposition 3.1,

$$sp(A) \subset N_\varepsilon\left(\bigcup_{\eta\in E} sp(A_\eta)\right).$$

So, we obtain

$$sp(A) \subseteq \bigcup_{\eta\in E} sp(A_\eta)$$

as ε tends to 0. □

4. Formula perturbation

This section is devoted to the study of pseudospectrum stability when the perturbation is applied directly to the operator's formula. For this purpose, we define for $\alpha > 0$,

$$A_\alpha = A_0 + \alpha V_3,$$

where V_3 is a continuous function over Ω such that

$$K_2 = \sup_{x\in\Omega} |V_3(x)| < \infty, K_3 = \inf_{x\in\Omega} |V_3(x)|,$$

which means that, for all $\alpha > 0$, $D(A_\alpha) = D(A_0)$. Our aim is to compare $sp(A_\alpha)$ and $sp(A_0)$. For $\eta > 0$, $A_{\alpha,\eta}$, $A_{0,\eta}$ are defined in the same way as A_η. It is clear, that for $\varepsilon > 0$,

$$\forall\alpha \geq 0, \quad sp_\varepsilon(A_\alpha) = \bigcup_{\eta\in E_\alpha} sp_\varepsilon(A_{\alpha,\eta}),$$

where $E_\alpha = \left\{\eta \in]0,1] \,:\, C_{PF}^{-2} + K - \frac{5K_1}{4} - \alpha K_3 \leq 0\right\}$, and

$$\forall\alpha \geq 0, \quad sp(A_\alpha) = \bigcup_{\eta\in E_\alpha} sp(A_{\alpha,\eta}).$$

Theorem 4.1.

$$\lim_{\alpha\to 0} sp(A_\alpha) \subset sp(A_0) \subset \lim_{\alpha\to 0} sp_{\alpha K_2}(A_\alpha).$$

P r o o f. Let $\varepsilon > 0$, $\eta > 0$ and $\lambda \in sp_\varepsilon(A_{0,\eta})$. Then there exists $f \in D(A_{0,\eta})$ such that

$$\|A_{0,\eta}f - \lambda f\|_{L^2(\Omega_\eta)} < \varepsilon \|f\|_{L^2(\Omega_\eta)}.$$

Therefore, for $\alpha > 0$,

$$\|A_{\alpha,\eta}f - \lambda f\|_{L^2(\Omega_\eta)} < (\varepsilon + \alpha K_2) \|f\|_{L^2(\Omega_\eta)},$$

which means that $\lambda \in sp_{\varepsilon+\alpha K_2}(A_{\alpha,\eta})$. However, $A_{0,\eta}$ and $A_{\alpha,\eta}$ are self-adjoint operators, then

$$sp(A_{0,\eta}) \subset sp_{\alpha K_2}(A_{\alpha,\eta}) \Rightarrow sp(A_0) \subset \bigcup_{\eta\in E_0} sp_{\alpha K_2}(A_{\alpha,\eta}).$$

We use the fact that $E_0 \subset E_\alpha$ for all $\alpha > 0$ to get

$$sp(A_0) \subset sp_{\alpha K_2}(A_\alpha) \Rightarrow sp(A_0) \subset \lim_{\alpha\to 0} sp_{\alpha K_2}(A_\alpha).$$

Inversely, it is clear that, for all $\alpha > 0$ and all $\eta > 0$,

$$sp(A_{\alpha,\eta}) \subset sp_{\alpha K_2}(A_{0,\eta}).$$

Then

$$sp\left(A_{\alpha}\right)=\bigcup_{\eta\in E_{\alpha}} sp\left(A_{\alpha,\eta}\right)\subset\bigcup_{\eta\in E_{\alpha}} sp_{\alpha K_2}\left(A_{0,\eta}\right)\Rightarrow\lim_{\alpha\to 0} sp\left(A_{\alpha}\right)\subset\lim_{\alpha\to 0}\bigcup_{\eta\in E_{\alpha}} sp_{\alpha K_2}\left(A_{0,\eta}\right),$$

but

$$\bigcup_{\eta\in E_{\alpha}} sp_{\alpha K_2}\left(A_{0,\eta}\right)=N_{\alpha K_2}\left(\bigcup_{\eta\in E_{\alpha}} sp\left(A_{0,\eta}\right)\right).$$

We use $\lim_{\alpha\to 0} E_{\alpha}=E_0$ to get

$$\lim_{\alpha\to 0} sp\left(A_{\alpha}\right)\subset sp\left(A_0\right).$$

□

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